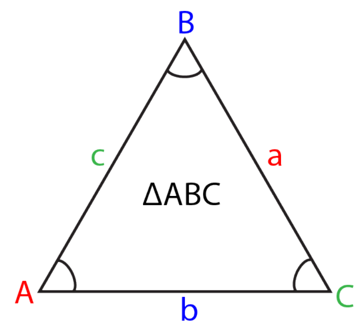
Grade 7/8 Math Circles
March 20/21/22/23, 2023
Trigonometry

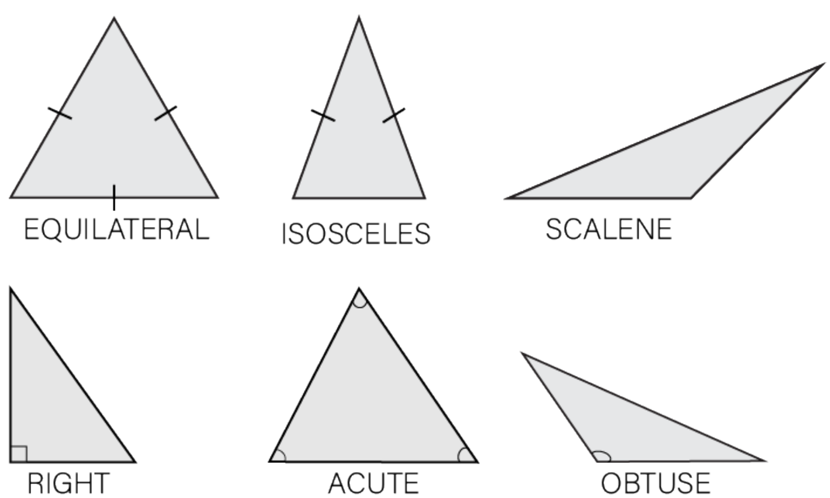
Triangles?

Triangles are 2-dimensional shapes with three sides. All of the angles in a triangle add up to 180°. On the right is an example of a convention used to label the sides and angles in this shape, using the first three letters of the alphabet. Notice that angle A is opposite of side a, angle B is opposite of side b, etc. The symbol Δ represents the word "triangle", and we name the triangle using the letters that represent all of the angles in it.



There are many different types of triangles, depending on the angles they have and the characteristics of their side lengths:

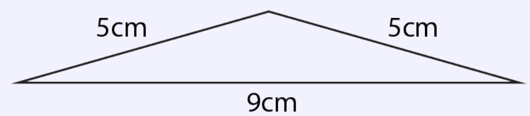
Equilateral: A triangle whose side lengths (and angles) are all equal.
Isosceles: A triangle with exactly two side lengths (and angles) that are equal.
Scalene: A triangle whose side lengths and angles are all different.
Right: A triangle that has a 90° angle.
Acute: A triangle whose angles are all less than 90°.
Obtuse: A triangle that has an angle greater than 90°.



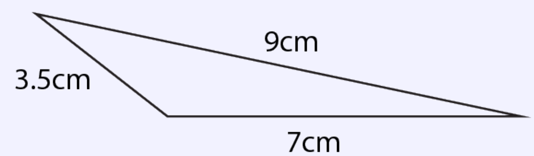
Exercise 1

Label the following triangles using the above terms. More than one can be applied!

a) Triangle A



b) Triangle B



Exercise 1 Solution

a) Triangle A is an obtuse, isosceles triangle.

b) Triangle B is an obtuse, scalene triangle.

As with most things in math, we like to go a bit more past the surface and really look at what makes triangles so special. If you aren't convinced yet that there is much more than meets the eye, I hope I will convince you by the end of this lesson. Let's see what mathematicians have had to say:

The field of mathematics concerned with triangles is called **trigonometry**. More formally, it is the study of angles and their relationship with ratios of side lengths in a triangle. The problems we encounter involve solving for some *unknown variable*, whether it be a side length or an angle.

Stop and Think

Where do you see triangles in your everyday life?

Right Triangles

Although there is so much to discover regarding these three-sided shapes, today we will be focusing solely on right triangles and what we can learn from them. Believe it or not, there is an abundance of knowledge to discover!

The **hypotenuse** is the longest side of a right triangle, also identified as being the side opposite of the $90°$ angle. It is most often labelled as being side c, where the other two sides are referred to as a and b. Below is a theorem you may already be familiar with, that can be very useful when trying to determine the side lengths of a triangle:

Pythagorean Theorem

For every right triangle with side lengths a, b, c, the following is always true:

$$a^2 + b^2 = c^2,$$

where c is the side length of the hypotenuse.

Exercise 2

Determine which of the following triangles can be a right triangle:

a) Triangle A with side lengths of 3cm, 5cm and 6cm

b) Triangle B with side lengths of 12ft, 9ft and 15ft

Exercise 2 Solution

a) This is *not* a right triangle since it does not satisfy the Pythagorean Theorem:

$$\begin{aligned} a^2 + b^2 &= 3^2 + 5^2 \\ &= 9 + 25 \\ &= 34 \end{aligned}$$

$$\text{but } c^2 = 6^2 = 36 \neq 34$$

b) This is indeed a right triangle:

$a^2 + b^2 = 9^2 + 12^2$ $\quad$ $c^2 = 15^2$

$a^2 + b^2 = 81 + 144$ $\quad$ $c^2 = 225$

$a^2 + b^2 = 225$

$\therefore a^2 + b^2 = c^2$

Stop and Think

Numbers that we know to satisfy the Pythagorean Theorem are called **Pythagorean Triples**. How many can you come up with?

Example A

A right triangle has side lengths of 4.2cm and 6.4cm. Determine the length of the third side, to the nearest tenth of a centimetre, if:

a) The third side *is* the hypotenuse.

If the third side is the hypotenuse, then 4.2 and 6.4 are our a and b values in the Pythagorean Theorem. Thus:

$$\begin{aligned} 4.2^2 + 6.4^2 &= c^2 \\ 17.64 + 40.96 &= c^2 \\ 58.6 &= c^2 \\ \sqrt{58.6} &= c \\ 7.655 &\approx c \end{aligned}$$

$\therefore$ the length of the third side is approximately 7.7cm.

b) The third side *is not* the hypotenuse.

If the third side is not the hypotenuse, then 6.4cm is the length of the longest side and is thus our c value in the Pythagorean Theorem. This means that we can label 4.2 as a and solve for b:

$$\begin{aligned} 4.2^2 + b^2 &= 6.4^2 \\ 17.64 + b^2 &= 40.96 \\ b^2 &= 40.96 - 17.64 \\ b^2 &= 23.32 \\ b &= \sqrt{23.32} \\ b &\approx 4.829 \end{aligned}$$

$\therefore$ the length of the third side is approximately 4.8cm.

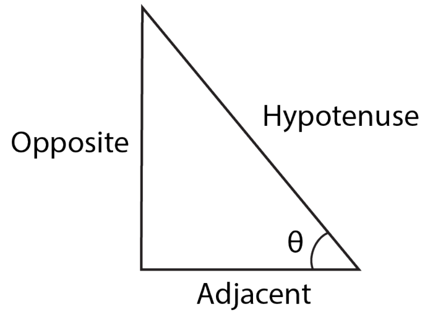
New Symbol Alert! $\approx$ is used when we have a non-terminating decimal and write down a rounded number to represent it.

Trigonometric Ratios

Recall the following definition from a previous week's lesson on Recursion:

"**Function:** a set of instructions that takes an input and produces an output."

There are six important functions that we use in higher mathematics which are directly tied to trigonometry. These are: $\sin(x), \cos(x), \tan(x), \csc(x), \sec(x), \cot(x)$ where x is some angle. We use these functions to help us to *solve a triangle*. This means that we have a missing side length or angle (or both!) that we want to determine. From now on, we will refer to a missing or known angle as θ, and label the sides of the triangle **in relation to** θ as follows:

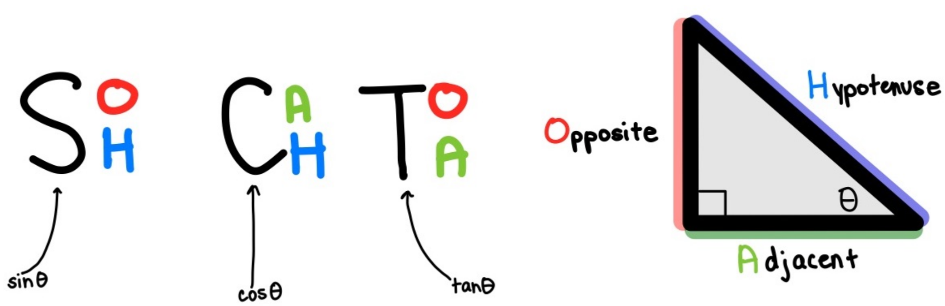


The hypotenuse will always be the longest side, but we label the other sides depending where they are in relation to θ: in the above picture, the vertical side is opposite θ so it is labelled "opposite". The horizontal side is adjacent (connected) to θ, so it is labelled "adjacent". In a bit, you will see that these sides are shortened to just be 'O', 'A' and 'H' for opposite, adjacent, and hypotenuse, respectively.

Why do we care about a jumble of three-letter "words" with a greek symbol? Well, when we **input** the angle θ into the functions, the **output** ends up being equal to the **ratio** of two of the sides!

sine function:	$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$	**cosecant function:**	$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$
cosine function:	$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$	**secant function:**	$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$
tangent function:	$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$	**cotangent function:**	$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$

A mnemonic that you may or may not have heard before to help us remember these ratios is SOH CAH TOA (pronounced: 'soe-kah-toe-ah') where 'S', 'C', 'T' stand for the sin, cos and tan functions, respectively:



You may notice that this mnemonic does not cover the ratios for csc, sec, and cot, and that they do not appear on your calculators. That is because these three are *reciprocals* of sin, cos, and tan, respectively. This means that all we have to do is "flip" the fraction of these three ratios to get the ratios of the other ones!

$\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

Also notice how the sine function tells you the height of the triangle above as a percentage of the hypotenuse, the cosine function tells you the length as a percentage of the hypotenuse, and the tangent function tells you the height as a percentage of the length. Given this, upon being told the value of one of these functions, you could visualize the triangle by comparing these ratios.

Exercise 3

Write a formula for $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Exercise 3 Solution

From our definitions, we know that $\sin\theta = \frac{O}{H}$, $\cos\theta = \frac{A}{H}$ and $\tan\theta = \frac{O}{A}$.

We want to use the fractions $\frac{O}{H}$ and $\frac{A}{H}$ to "create" the fraction $\frac{O}{A}$, which would mean cancelling out the H in the denominators. Using knowledge of fraction division we can try the following:

$$\frac{O}{H} \div \frac{A}{H} = \frac{O}{H} \times \frac{H}{A} = \frac{O}{A} = \tan\theta$$

So we can write tan as a ratio of sin over cos, or: $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Example B

Consider a triangle whose height is 3cm, and has an angle of $30°$ opposite of the height. The sine trigonometric ratio tells us that $\sin(30°) = 0.5$. This tells us that the height is 50% of the hypotenuse, and we could then say that our hypotenuse is $3 \div 0.5 = 6$ centimetres long.

The tangent trigonometric ratio tells us that $\tan(30°) \approx 0.5774$, so the height of the triangle is a bit more than half of the width of the triangle (aka the adjacent side). Without even seeing the triangle, we can approximate the width to be about $3 \div 0.5774 \approx 5.196$ centimetres long.

Stop and Think

What information do the cosecant, secant, and cotangent functions tell you about your height, length and hypotenuse, in terms of percentages of each other?

I could tell you to accept these to be functions that happen to work perfectly with right triangles, but I can also propose an example so you can visualize what is going on for them to work:

Imagine that you were standing in a big movie theatre with an arched ceiling, where the highest point is 1 decimeter. You want to hang up a screen that hangs from the ceiling to the floor, but cannot seem to decide where you should hang it: at the edge of the room so that it is closer to the ground, in the very middle so that the screen is as high as it can be, or somewhere in between. Note that the distance from the center of the floor to the top of the screen will always stay the same, since it is a circular dome. The image below may help you visualize:

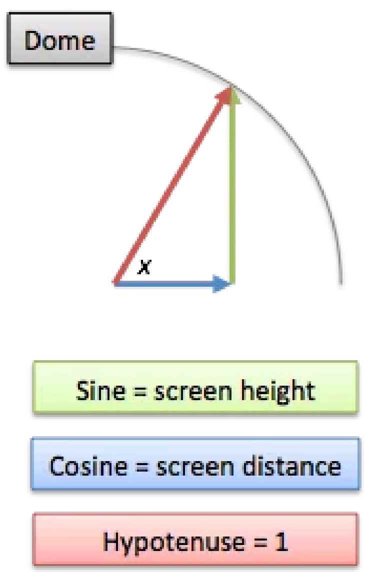


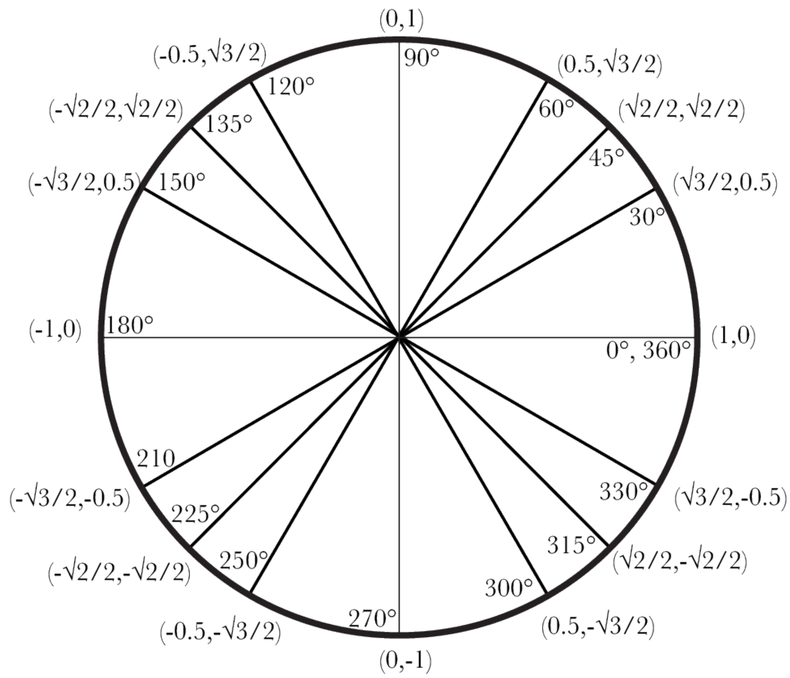
Image taken from Better Explained

If you were to determine where you hang your screen based off of the angle from the floor to the top of the screen (x), you could easily calculate its height using $\sin(x)$ or $\cos(x)$.

Now that we have a general idea of how these trigonometric functions work, we can look at them a bit more mathematically:

The **Unit Circle** is a circle with a radius of 1, whose coordinates are determined by $\sin\theta$ and $\cos\theta$.

Further, each coordinate makes an angle with the x-axis when a line is drawn from the point to the origin. If we then draw a straight vertical line from the point to the x-axis, we get a right triangle!



Example C

Determine the value of the following trigonometric ratios using the Unit Circle. Round to four decimal places if necessary:

$\sin(30°) = 0.5$

$\cos(30°) = 0.8660$

$\sin(45°) = 0.7071$

$\cos(45°) = 0.7071$

$\sin(120°) = 0.8660$

$\cos(120°) = -0.5$

Notice how we did not *really* have to use SOH CAH TOA to solve for the sin and cos ratios, since the hypotenuse is 1 and anything divided by 1 is just itself!

Exercise 4

Try calculating $\tan(90°)$ and $\tan(270°)$. What did you get? Why does this make sense?

Exercise 4 Solution

Plugging in $\tan(90°)$ and $\tan(270°)$ into a calculator gives an error message (depending on your calculator, it will say "Error" or "Math Error"). This indeed makes sense because of our unit circle: at $\theta = 90°$, $\sin\theta = 1$ and $cos\theta = 0$. Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\tan(90°) = \frac{1}{0}$. But this fraction is *undefined* because you can't divide by 0!

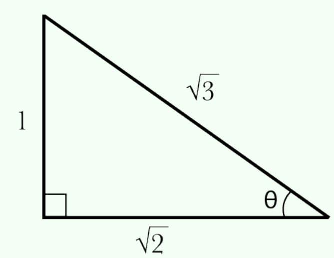
Example D

A right-angled triangle has side lengths of 1 cm, $\sqrt{2}$ cm and $\sqrt{3}$ cm as shown. We want to determine the three primary trigonometric ratios for the angle θ:

$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{3}}$

$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{2}}{\sqrt{3}}$

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{2}}$



What if we wanted to figure out what θ actually was? Can we do this?

... indeed we can! Our handy dandy calculators have a button for the *inverse functions* of our trigonometric functions. You can think of it as "undoing" our function steps. Instead of having θ as the input and our ratio as the output, the inverses take our ratio as the input and give θ as the output. Notice how we only need to choose one of $\sin, \cos, \tan$ since they all give the same value for θ!

$\sin^{-1}(\frac{1}{\sqrt{3}}) \approx 35.3°$

$\cos^{-1}(\frac{\sqrt{2}}{\sqrt{3}}) \approx 35.3°$

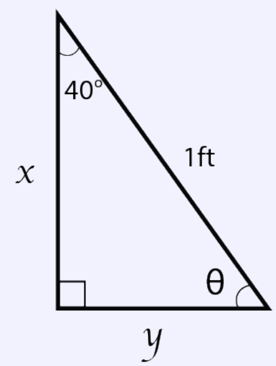
$\tan^{-1}(\frac{1}{\sqrt{2}}) \approx 35.3°$

Now that we have a bit of practice using our trigonometric functions, let's see if we can apply them to solve for triangles like I hinted at before.

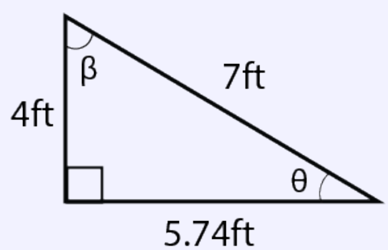
Exercise 5

Solve for all the missing sides and angles of the triangles below:

1. Triangle 🦇



2. Triangle ❤



Exercise 5 Solution

1. Since we are given two of three angles, we can solve for θ using the sum of angles in a triangle:

$$180^\circ = 40^\circ + 90^\circ + \theta$$
$$180^\circ - 40^\circ - 90^\circ = \theta$$
$$50^\circ = \theta$$

We can now use our trigonometric ratios to solve for x and y:

$$\sin(50°) = \frac{x}{1} \qquad \cos(50°) = \frac{y}{1}$$
$$\sin(50°) = x \qquad \cos(50°) = y$$
$$0.7660 \approx x \qquad 0.6428 \approx y$$

2. In this triangle we are given all of the sides and need to determine two angles, so we need to use the inverse of our trigonometric ratios:

$$\theta = \sin^{-1}(\frac{4}{7}) \qquad \beta = \cos^{-1}(\frac{4}{7})$$
$$\theta \approx 34.85° \qquad \beta \approx 55.15°$$

Trigonometric Identities

An **identity** is an equation which asserts that two expressions are *always* equal. One of the most famous trigonometric identities is the **Pythagorean Identity** which states that for all angles θ, $\sin^2\theta + \cos^2\theta = 1$.

Exercise 6

Prove the Pythagorean Identity.

Hint: Look carefully at the name of this identity, and use the unit circle

Exercise 6 Solution

As we saw in the lesson, for any angle θ, the point on the Unit Circle where the terminal arm of the angle intersects it is $(\cos\theta, \sin\theta)$. Further, if we draw a vertical line down from the point, it creates a right triangle with the x-axis. Since the hypotenuse = radius = 1, the "height" of the triangle is $\sin\theta$ and the "width" of the triangle is $\cos\theta$.

We can apply the Pythagorean Theorem to the sides of the right triangle we created:

$$a^2 + b^2 = c^2$$
$$(\sin\theta)^2 + (\cos\theta)^2 = 1^2$$
$$\sin^2\theta + \cos^2\theta = 1$$

Thus we have proven the Pythagorean Identity.

We can now make a list of all the different identities and ratios that we have learned:

$\sin^2\theta + \cos^2\theta = 1$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\csc\theta = \frac{1}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$

$\cot\theta = \frac{1}{\tan\theta}$

Example E

We will try to prove the following:

$$\cos\theta = \cot\theta \times \sin\theta$$

When proving identities, we usually look at the "left side" (LS) and the "right side" (RS) separately, trying to either simplify one side to look like the other, or simplify both sides into something more concise.

Since the LS is fairly simple, we will try to simplify the RS to look like the LS:

LS	RS
$\cos\theta$	$\cot\theta \times \sin\theta$
	$= \frac{\cos\theta}{\sin\theta} \times \sin\theta$
	$= \cos\theta$

$\therefore$ LS = RS, and so we have proven that $\cos\theta = \cot\theta \times \sin\theta$

Exercise 7

Prove the following:

a) $\sec\theta - \tan\theta\sin\theta = \frac{1}{\sec\theta}$

b) $\frac{1+\cos\theta}{\sin\theta} = \csc\theta + \cot\theta$

Exercise 7 Solution

a)

LS	RS
$\sec\theta - \tan\theta\sin\theta$	$\frac{1}{\sec\theta}$
$= \frac{1}{\cos\theta} - \tan\theta\sin\theta$	$= \cos\theta$
$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\sin\theta$	
$= \frac{1-\sin^2\theta}{\cos\theta}$	
$= \frac{\cos^2\theta}{\cos\theta}$	
$= \cos\theta$	

$\therefore$ LS = RS, and so we have proven that $\sec\theta - \tan\theta\sin\theta = \frac{1}{\sec\theta}$.

b)

LS	RS
$\frac{1+\cos\theta}{\sin\theta}$	$\csc\theta + \cot\theta$
$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$	
$= \csc\theta + \frac{\cos\theta}{\sin\theta}$	
$= \csc\theta + \cot\theta$	

$\therefore$ LS = RS, and so we have proven that $\frac{1+\cos\theta}{\sin\theta} = \csc\theta + \cot\theta$.